

A Rotation Matrix

First we note that

$$K^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$K^3 = K^2 K = -K, K^4 = K^2 K^2 = I, K^5 = K^4 K = K, \text{ etc.}$$

Therefore

$$\begin{aligned} e^{\theta K} &= I + \theta K + \frac{1}{2!} \theta^2 K^2 + \frac{1}{3!} \theta^3 K^3 + \frac{1}{4!} \theta^4 K^4 + \frac{1}{5!} \theta^5 K^5 + \dots \\ &= I + \theta K + \frac{1}{2!} \theta^2 (-I) + \frac{1}{3!} \theta^3 (-K) + \frac{1}{4!} \theta^4 (I) + \frac{1}{5!} \theta^5 (K) + \dots \\ &= \left(I - \frac{1}{2!} \theta^2 + \frac{1}{4!} \theta^4 - \dots \right) I + \left(\theta - \frac{1}{3!} \theta^3 + \frac{1}{5!} \theta^5 - \dots \right) K \\ &= (\cos \theta) I + (\sin \theta) K = \begin{bmatrix} \cos \theta & 0 \\ 0 & \cos \theta \end{bmatrix} + \begin{bmatrix} 0 & -\sin \theta \\ \sin \theta & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad (\text{i.e. a } \theta \text{ radians rotation of the plane}) \end{aligned}$$