Differential Systems Revisited

In Part 1 of this topic we solved a system of differential equations by looking for exponential solutions. We take now a different but related approach. If a and x are real numbers and $u(t)$ is a real-valued function, then the solution of a simple initial value problem in one unknown is

$$\frac{du}{dt} = au, u(0) = x \Rightarrow u(t) = xe^{at}$$

This we know from calculus I (use the separation of variables method). If instead x is in $\mathbb{R}^n$, $u(t)$ is an $\mathbb{R}^n$-valued function and A is an $n \times n$ matrix, then

$$\frac{du}{dt} = Au, u(0) = x$$

is the initial value problem for a linear system of n differential equations in n unknowns. The goal here is to show that its solution is (similarly)

$$u(t) = e^{tA}x$$

provided we can define the exponential of a square matrix in an appropriate way.