An Example with Complex Eigenvalues and Eigenvectors

Ex: Consider $K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$ which is a 90 degree rotation of the plane. We have

$$0 = \det(K - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 \Rightarrow \lambda = \pm i$$

For the eigenvectors we have:
Case 1, $\lambda = i$:

$$(K - iI)x = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} -ia - b = 0 \\ a - ib = 0 \end{array}$$

If we multiply the second equation by $-i$ (yes, we must now allow scalars that are complex numbers) we obtain the first equation and in this sense $K - iI$ is singular. Thus $a = ib$. If we set $a = i$ we get $b = 1$ and this gives us the eigenvector

$$x = \begin{bmatrix} i \\ 1 \end{bmatrix}$$