Euler's Identity

There is an important identity that allows us to represent complex numbers in terms of their polar coordinates in the complex plane. This is **Euler's identity**. Start with the power series expansion $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$. Then

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{1}{2!}i^2\theta^2 + \frac{1}{3!}i^3\theta^3 + \frac{1}{4!}i^4\theta^4 + \frac{1}{5!}i^5\theta^5 + \cdots \\
&= 1 + i\theta + \frac{1}{2!}(-1)\theta^2 + \frac{1}{3!}(-i)\theta^3 + \frac{1}{4!}(1)\theta^4 + \frac{1}{5!}(i)\theta^5 + \cdots \\
&= (1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \cdots) + i(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots) \\
&= \cos\theta + i\sin\theta
\end{aligned}$$

This shows that $e^{i\theta}$ is indeed a complex number (since we can see its real and imaginary parts).