Cramer's Rule

Now we easily see that $\det A = 2(3) - 1(2) + 0(1) = 4$ and therefore

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Theorem: (Cramer's rule) Let $B_i = \begin{bmatrix} a_1 & a_2 & \cdots & b & \cdots & a_n \end{bmatrix}$ be the matrix A but with column i replaced by the vector b. Then the solution x of $Ax = b$ has entries

$$x_i = \frac{\det B_i}{\det A}, i = 1, ..., n$$

Proof: Here is the 3×3 case. If $b = Ax$, then $b = x_1 a_1 + x_2 a_2 + x_3 a_3$. Therefore

$$\begin{aligned} \det B_1 &= \begin{bmatrix} x_1 a_1 + x_2 a_2 + x_3 a_3 & a_2 & a_3 \end{bmatrix} \\ &= \begin{bmatrix} x_1 a_1 & a_2 & a_3 \end{bmatrix} \text{ (PROP 5 for columns)} \\ &= x_1 \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \text{ (DEF PROP 3 for columns)} = x_1 \det A \end{aligned}$$