The Adjoint and Inverses

Let $C = [C_{ij}]$ be the matrix of cofactors of A. We call C the **adjoint** of A.

Theorem: If $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A} C^T$$

This is the generalization to $n \times n$ matrices of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Indeed, for this 2×2 matrix, $C_{11} = d$, $C_{12} = -c$, $C_{21} = -b$, $C_{22} = a$.
Proof of theorem:

$$AC^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \det A \end{bmatrix}$$