More Observations

Consider the matrix

$$M_{ij} = \left[\begin{array}{ccc|c|ccc} & A_{11} & & \begin{matrix} a_{1j} \\ \vdots \\ a_{i-1,j} \end{matrix} & & A_{12} & \\ \hline 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \hline & A_{21} & & \begin{matrix} a_{i+1,j} \\ \vdots \\ a_{nj} \end{matrix} & & A_{22} & \end{array}\right]$$

Row i here is row j of the identity matrix. A_{11} is the upper left $(i-1)\times(j-1)$ part of A, A_{22} is the lower right $(n-i)\times(n-j)$ part of A, etc. If we remove row i and column j of M_{ij}, we get A with its row i and column j removed.

Linearity of $\det A$ in its ith row gives us

$$|A| = a_{i1}\,|M_{i1}| + a_{i2}\,|M_{i2}| + \cdots + a_{in}\,|M_{in}|$$