Some Determinant Expansions

Here we want to consider two ways of computing determinants: by permutations and by cofactor expansions.

Consider first 2×2 matrices and use linearity in rows 1 and 2 along with

$$\begin{bmatrix} a & b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} c & d \end{bmatrix} = c\begin{bmatrix} 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \end{bmatrix}$$

This gives

$$\begin{aligned} \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= a\begin{vmatrix} 1 & 0 \\ c & d \end{vmatrix} + b\begin{vmatrix} 0 & 1 \\ c & d \end{vmatrix} \\ &= ac\underbrace{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}_{\text{zero column}} + ad\underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{\text{det of } I} + bc\underbrace{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}_{\text{permutation}} + bd\underbrace{\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}}_{\text{zero column}} \\ &= ad - bc \end{aligned}$$