

Some Additional Properties

$$\begin{aligned}
 &= a_{33} \begin{vmatrix} 0 & a_{22} & 0 \\ 0 & 0 & 1 \\ a_{11} & a_{12} & 0 \end{vmatrix} \underbrace{= a_{22}a_{33}}_{\text{DEF PROP 3}} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{11} & a_{12} & 0 \end{vmatrix} \underbrace{= a_{22}a_{33}}_{\text{PROP 5}} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{11} & 0 & 0 \end{vmatrix} \\
 &\quad \underbrace{\text{PROP 5 (twice)}} \\
 &= a_{22}a_{33} \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \underbrace{= a_{11}a_{22}a_{33}}_{\text{DEF PROP 3}} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \underbrace{= a_{11}a_{22}a_{33}}_{\text{DEF PROP 1}} \\
 &\quad \underbrace{\text{DEF PROP 2 (twice)}} \quad \underbrace{\text{DEF PROP 3}}
 \end{aligned}$$

Ex:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underbrace{\begin{vmatrix} a & b \\ 0 & d - \frac{c}{a} \end{vmatrix}}_{\text{PROP 5}} = \underbrace{a \left(d - \frac{c}{a} \right)}_{\text{PROP 7}} = ad - bc \quad (\text{assuming } a \neq 0)$$