Functions as Vectors

If $f(t)$ and $g(t)$ are functions, we know how to add these functions together and multiply either by a real number. Thus sets of functions are vector spaces. Consider a function $f(t)$ defined on an interval $[a, b]$. How can we think of f as a vector? Partition $[a, b]$ into n equal divisions and replace f by its values at the points in the partition:

$$t_0 = a < t_1 < t_2 < \cdots < b = t_n, \Delta t_i = \frac{b-a}{n}$$

$$f \approx \begin{bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_n) \end{bmatrix} \quad \text{(approximately for } n \text{ large)}$$

This suggests that the "length" or "size" of f should be

$$\|f\|^2 = f(t_0)^2 + f(t_1)^2 + \cdots + f(t_n)^2 = \sum_{i=0}^{n} f(t_i)^2$$

Unfortunately this number gets larger and larger as n gets larger.