Permutation Matrices Are Orthogonal

Ex: Permutation matrices are orthogonal! Let's illustrate this in $\mathbb{R}^3$. Let the columns of $I = I_3$ be e_1, e_2, e_3 and let P be the matrix that moves row 1 to 2, 2 to 3, and 3 to 1. Then by a simple calculation:

$$P = \begin{bmatrix} - & e_3^T & - \\ - & e_1^T & - \\ - & e_2^T & - \end{bmatrix} \Rightarrow PP^T = \begin{bmatrix} - & e_3^T & - \\ - & e_1^T & - \\ - & e_2^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ e_3 & e_1 & e_2 \\ | & | & | \end{bmatrix} = I$$

(why?) and so $P^{-1} = P^T$.

Theorem: Let Q be orthogonal. Then for any u and v in $\mathbb{R}^n$

$$\|Qu\| = \|u\| \quad \text{(lengths are preserved)}$$
$$(Qu, Qv) = (u, v) \quad \text{(inner products are preserved)}$$

Proof: $(Qu, Qv) = (Qu)^T Qv = u^T Q^T Qv = u^T v = (u, v)$. The first conclusion follows from this one by setting $u = v$.

Geometrically rotations and reflections maintain lengths and angles (inner products), as does any composition of the two.