Linear Systems with Orthonormal Columns

$$= \begin{bmatrix} (q_1, q_1) & \cdots & (q_1, q_k) \\ \vdots & \ddots & \vdots \\ (q_k, q_1) & \cdots & (q_k, q_k) \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = I$$

Conclusion: If Q has orthonormal columns, then $Q^T Q = I$.

Ex: Try this out for yourself in $\mathbb{R}^3$ with $Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

So what is the value of these observations? Consider $Ax = b$ again: is b in the column space of A? Let $q_1, ..., q_k$ be an orthonormal basis of $C(A)$ and Q be the matrix with this basis as columns. Then $C(A) = C(Q)$ (why?). Then

$$Ax = b \iff Qy = b \iff \underbrace{Q^T Qy = Q^T b}_{\text{normal equations}} \iff y = Q^T b \text{ (wow!)}$$