Orthonormal Bases (continued)

Then our general projection onto S has the form

$$Pv = AA^T v = \begin{bmatrix} | & \cdots & | \\ q_1 & \ddots & q_r \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} (q_1, v) \\ \vdots \\ (q_r, v) \end{bmatrix} = (q_1, v)q_1 + \cdots + (q_r, v)q_r$$

Some rearrangement gives

$$Pv = (q_1^T v)q_1 + \cdots + (q_r^T v)q_r = (q_1 q_1^T)v + \cdots + (q_r q_r^T)v$$

That is:

$$P = q_1 q_1^T + \cdots + q_r q_r^T$$

Theorem: The projection onto a subspace S of $\mathbb{R}^n$ is the sum of the projections onto the members of an othonormal basis of S.