Some Observations

The preceding calculations involved $(A^T A)^{-1}$ and so it is important to know when $A^T A$ is invertible. It turns out this is the case whenever the columns of A are linearly independent. We establish this fact through a series of observations:

- A has linearly independent columns if and only if $N(A) = \{0\}$ (why?)
- if B is square, then B is invertible if and only if $N(B) = \{0\}$ (why?)
- A and $B = A^T A$ have the same null space:
  1. x in $N(A) \Rightarrow Ax = 0 \Rightarrow A^T Ax = 0 \Rightarrow x$ is in $N(A^T A)$
  2. x in $N(A^T A) \Rightarrow A^T Ax = 0 \Rightarrow x^T A^T Ax = 0 \Rightarrow (Ax)^T (Ax) = 0 \Rightarrow \|Ax\|^2 = 0 \Rightarrow Ax = 0 \Rightarrow x$ is in $N(A)$
- columns of A linearly indep
  $\Rightarrow N(A) = \{0\} \Rightarrow N(A^T A) = \{0\} \Rightarrow B = A^T A$ is invertible