Projecting a Vector Onto a Subspace

Let A be an $m \times n$ matrix and b a vector NOT in the column space of A. Then $Ax = b$ is inconsistent. We want to replace this system with a new one $A\hat{x} = p$ where p IS in the column space of A. The error in using p rather than b is

$$e = b - p = b - A\hat{x}$$

We want e to be orthogonal to the subspace $C(A)$ (to get the smallest error), and therefore e is in $C(A)^{\perp}$. But this is the same as saying that e is in $N(A^T)$, i.e. $A^T e = 0$:

$$0 = A^T(b - A\hat{x}) = A^T b - A^T A\hat{x}$$

$$\Rightarrow \boxed{A^T A\hat{x} = A^T b}$$

These are called the **normal equations** corresponding to $Ax = b$ and define the **projection** $p = A\hat{x}$ **of** b **onto the column space of** A.