Projection Matrix

Definition: The vector

$$p = \frac{(a, b)}{\|a\|^2} a$$

is called the orthogonal projection of the vector b onto the vector a. It is the vector closest to b but lying on the line formed by a, closest in the send of minimum squared error $\|e\|^2 = \|b - p\|^2$.

There is an alternate way to express a projection, namely as the result of multiplying b by a "projection matrix." Note:

$$(a, b)a = \begin{bmatrix} (a_1 b_1 + \cdots + a_n b_n)a_1 \\ \vdots \\ (a_1 b_1 + \cdots + a_n b_n)a_1 \end{bmatrix} = \begin{bmatrix} a_1^2 b_1 + \cdots + a_1 a_n b_n \\ \vdots \\ a_1 a_n b_1 + \cdots + a_n^2 b_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1^2 & \cdots & a_1 a_n \\ \vdots & \ddots & \vdots \\ a_n a_1 & \cdots & a_n^2 \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \right) b$$