Unique Representations of Vectors

When we have a subspace S of $\mathbb{R}^n$, we typically write

$$\mathbb{R}^n = S \oplus S^{\perp}$$

to mean that any vector v can be written **uniquely** as the sum of a vector v_S in S and a vector $v_{S^{\perp}}$ in $S^{\perp}$. Thus, for example,

$$\mathbb{R}^n = C(A^T) \oplus N(A), \mathbb{R}^m = C(A) \oplus N(A^T)$$

Here is how we find the two vectors. Find a basis $\{s_1, s_2, ..., s_r\}$ of S and a basis $\{s_{r+1}, s_{r+2}, ..., s_n\}$ of $S^{\perp}$. Then by independence with in each set and orthogonality across the two sets, $\{s_1, ..., s_r, s_{r+1}, ..., s_n\}$ is linearly independent and therefore a basis of $\mathbb{R}^n$. Therefore there is a unique representation:

$$v = \underbrace{c_1 s_1 + \cdots + c_r s_r}_{v_S} + \underbrace{c_{r+1} s_{r+1} + \cdots + c_n s_n}_{v_{S^{\perp}}}$$