Finding Orthogonal Complements

In general, if S and W are subspaces of $\mathbb{R}^n$ and $\dim S + \dim W = n$, then $W = S^\perp$ and $S = W^\perp$. Recalling that $\dim C(A) = r$, $\dim C(A^T) = r$, $\dim N(A) = n - r$, and $\dim N(A^T) = m - r$, we conclude that

$$N(A)^\perp = C(A^T), C(A)^\perp = N(A^T), N(A^T)^\perp = C(A), C(A^T)^\perp = N(A)$$

Given S, how do you go about find $S^\perp$? The result above provides the clue. Let $\{s_1, s_2, ..., s_r\}$ be a basis of S and let A be a matrix with the basis vectors as columns. Then $C(A)$ is the set of all linear combos of the basis vectors, that is $S = C(A)$. Therefore $S^\perp = N(A^T)$.

Ex: Let

$$S = \text{Span}\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \}$$

Find a basis of $S^\perp$.