Orthogonal Subspaces

Definition: Let S and U be subspaces of a vector space V. We say that S is **orthogonal** to U, denoted $S \perp U$, if $(s, u) = 0$ for every vector s in S and every vector u in U (everything in S is orthogonal to everything in U).

Note: Even though our inner product is only defined at this time for vectors in $\mathbb{R}^n$, later we will introduce inner products for other vector spaces and the same definitions and results will apply there.

In Topic 2 Part 4 we saw an example in $\mathbb{R}^2$ that suggested that certain of the fundamental subspaces are orthogonal to each other. Let us verify that fact now:

Theorem: Let A be $m \times n$. Then $N(A^T) \perp C(A)$ (in $\mathbb{R}^m$) and $N(A) \perp C(A^T)$ (in $\mathbb{R}^n$)

Proof: Take any y in $N(A^T)$. This means that $A^T y = 0$. Now take any b in $C(A)$. This means that $b = Ax$ for some x. Then

$$(y, b) = y^T b = y^T Ax = (A^T y)^T x = 0^T x = 0$$

The second conclusion follows from the first: replace A with A^T.