A Feature of Orthonormal Bases

Here is one very important feature that orthonormal bases possess:

Theorem (coordinates relative to an orthonormal basis):
If

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

where $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis, then

$$c_i = (v_i, v), i = 1, 2, \ldots, n$$

Proof: Take the inner product of both sides with v_1:

$$(v_1, v) = c_1 \underbrace{(v_1, v_1)}_{1 \text{ (normal vec)}} + c_2 \underbrace{(v_1, v_2)}_{0 \text{ (orthogonal)}} + \cdots + c_n \underbrace{(v_1, v_n)}_{0 \text{ (orthogonal)}}$$

which gives $c_1 = (v_1, v)$. The other coefficients are done in the same way.

So expressing a vector as a linear combo of an orthonormal basis is easy! Just compute some inner products.