A Feature of Mutual Orthogonality

Here is one very important feature that orthogonality gives us:

Theorem: Mutual orthogonality implies linear independence.
Proof: Let $v_1, v_2, \ldots, v_k$ be mutually orthogonal non-zero vectors and suppose that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

Take the inner product of each side with v_1:

$$\begin{aligned}
(c_1 v_1 + c_2 v_2 + \cdots + c_k v_k, v_1) &= (0, v_1) = 0 \\
c_1(v_1, v_1) + c_2(v_2, v_1) + \cdots + c_k(v_k, v_1) &= 0 \text{ (linear in first argument)} \\
c_1 \|v_1\|^2 + c_2 0 + \cdots + c_k 0 &= 0 \text{ (mutual orthogonality)} \\
c_1 &= 0 \text{ (since } v_1 \neq 0\text{)}
\end{aligned}$$

In the same way we can show that the other c_i's are all zero.