An Example

Ex: For the S in the preceding example, find the unique representation of $v = \begin{bmatrix} 1 & 3 & 9 & 5 \end{bmatrix}^T$. In this case we already have bases of S and $S^\perp$ and we just use them together as a basis of $\mathbb{R}^4$:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 8 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

Therefore

$$\begin{bmatrix} 1 \\ 3 \\ 9 \\ 5 \end{bmatrix} = \underbrace{2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{v_S} \underbrace{+4\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_{S^\perp}} = \underbrace{\begin{bmatrix} 5 \\ 3 \\ 5 \\ 0 \end{bmatrix}}_{v_S} + \underbrace{\begin{bmatrix} -4 \\ 0 \\ 4 \\ 5 \end{bmatrix}}_{v_{S^\perp}}$$