Lengths and Inner Products

We now have the general structure of linear algebra in place and we want to add some extra features. Here we add orthogonality.

Definition: For x and y in $\mathbb{R}^n$:

$$\|x\| = \text{ (\textbf{length} of } x) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \text{ (Pythagorean Thm)}$$
$$(x, y) = \text{ (\textbf{inner product} of } x \text{ and } y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Note that $(x, x) = x_1^2 + x_2^2 + \cdots + x_n^2 = \|x\|^2$. The most important properties of the inner product, easily verified here, are $(x, y) = (y, x)$ (symmetric) and $(c_1 x_1 + c_2 x_2, y) = c_1 (x_1, y) + c_2 (x_2, y)$ (linearity in the first argument). By symmetry we then also have linearity in the second argument. We note in passing that $(x, y) = x^T y$ (as used in our text).

Definition: We say x is a **unit vector** if $\|x\| = 1$. We say x and y are **orthogonal** if $(x, y) = 0$.