

Euler's Identity

Euler established an interesting identity concerning “connected” graphs:

$$(\# \text{ nodes}) - (\# \text{ edges}) + (\# \text{ small loops}) = 1$$

With our machinery we can easily establish this. If the graph is connected then the dimension of the null space is 1. This means the rank of A is $r = n - 1$ (nullity + rank = n). And the dimension of the left null space is $m - r$. Hence

$$\begin{aligned}(\# \text{ nodes}) - (\# \text{ edges}) + (\# \text{ small loops}) &= n - m + (m - r) \\ &= n - m + (m - (n - 1)) \\ &= 1\end{aligned}$$