Why does this rule work?

- First note that
$$10 \equiv 1 \pmod 9.$$
- This means that
$$10^k \equiv 1^k \pmod 9 = 1 \pmod 9.$$
- Now, if a $k+1$-digit number has digits
$$a_k, a_{k-1}, \dots, a_1, a_0,$$
meaning that it is the number
$$n = a_k 10^k + a_{k-1} 10^{k-1} + \dots + a_1 \cdot 10 + a_0 \cdot 1,$$
then
$$\begin{aligned} n &\equiv (a_k 10^k + a_{k-1} 10^{k-1} + \dots + a_1 \cdot 10 + a_0 \cdot 1) \pmod 9 \\ &\equiv (a_k \cdot 1 + a_{k-1} \cdot 1 + \dots + a_1 \cdot 1 + a_0 \cdot 1) \pmod 9 \\ &\equiv (a_k + a_{k-1} + \dots a_1 + a_0) \pmod 9. \end{aligned}$$
- Thus summing the digits of a number does not change its remainder modulo 9.