Another proof

- We have

$$G_X(t) = \sum_{k=0}^{\infty} t^k \mathbb{P}(X = k), \qquad G_Y(t) = \sum_{l=0}^{\infty} t^l \mathbb{P}(Y = l).$$

- Then

$$\begin{aligned} G_X(t) G_Y(t) &= \left(\sum_{k=0}^{\infty} t^k \mathbb{P}(X = k) \right) \left(\sum_{l=0}^{\infty} t^l \mathbb{P}(Y = l) \right) \\ &= \sum_{k,l=0}^{\infty} t^{k+l} \mathbb{P}(X = k) \mathbb{P}(Y = l). \end{aligned}$$

- Write $m = k + l$, and we can resum

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{k=0}^{m} t^m \mathbb{P}(X = k) \mathbb{P}(Y = m - k) &= \sum_{m=0}^{\infty} t^m \sum_{k=0}^{m} \mathbb{P}(X = k \wedge Y = m - k) \\ &= \sum_{m=0}^{\infty} t^m \sum_{k=0}^{m} \mathbb{P}(X + Y = m) = G_{X+Y}(t). \end{aligned}$$