Lemma (Chebyshev's Inequality)

Let X be random variable, with $\mathbb{E}[X] = \mu < \infty$. Then for any $\epsilon > 0$,

$$\mathbb{P}(|X - \mu| \geq \epsilon) \leq \frac{\mathsf{Var}(X)}{\epsilon^2}.$$

Proof

We have

$$\mathbb{P}(|X - \mu| \geq \epsilon) = \sum_{\omega : |X(\omega) - \mu| \geq \epsilon} p(\omega).$$

Then

$$\begin{aligned}
\mathsf{Var}(X) &= \sum_{\omega} |X(\omega) - \mu|^2 \, p(\omega) \geq \sum_{\omega : |X(\omega) - \mu| \geq \epsilon} |X(\omega) - \mu|^2 \, p(\omega) \\
&\geq \sum_{\omega : |X(\omega) - \mu| \geq \epsilon} \epsilon^2 p(\omega) = \epsilon^2 \sum_{\omega : |X(\omega) - \mu| \geq \epsilon} p(\omega) \\
&= \epsilon^2 \mathbb{P}(|X - \mu| \geq \epsilon).
\end{aligned}$$