Corollary

We have $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$*, with equality iff* $V(X) = 0$ *iff* $\mathbb{P}(X = \mu) = 1$.

Proof

- Note $V(X) \geq 0$ since $(X - \mu)^2 \geq 0$.
- Thus $\mathbb{E}[X^2] - (\mathbb{E}[X])^2 \geq 0$.
- Clearly we only obtain equality if $V(X)$.
- Now assume that $V(X) = \mathbb{E}[(X - \mu)^2] = 0$. We compute:

$$\begin{aligned}
&\mathbb{E}[(X - \mu)^2] \\
&= \mathbb{E}[(X - \mu)^2 | X = \mu]\mathbb{P}(X = \mu) + \mathbb{E}[(X - \mu)^2 | X \neq \mu]\mathbb{P}(X \neq \mu) \\
&= 0 \cdot \mathbb{P}(X = \mu) + (\text{positive}) \cdot \mathbb{P}(X \neq \mu).
\end{aligned}$$