Theorem

Let the sets $A_1, \ldots, A_n$ be disjoint, then

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mathbb{P}(A_k).$$

Proof.

If $n = 2$ this is what we proved earlier, so this is the base case. Assume true for n. Then

$$\bigcup_{k=1}^{n+1} A_k = \left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1} \text{ and } \left(\bigcup_{k=1}^{n} A_k\right) \cap A_{n+1} = \emptyset.$$

Therefore

$$\mathbb{P}\left(\bigcup_{k=1}^{n+1} A_k\right) = \mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) + \mathbb{P}(A_{n+1}),$$

and by the induction hypothesis, this is

$$\sum_{k=1}^{n} \mathbb{P}(A_k) + A_{n+1} = \sum_{k=1}^{n+1} \mathbb{P}(A_k).$$

□