

## Theorem

*The cardinality of any interval is the same as the whole real line.*

## Proof.

- Let us define

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

- domain:  $\mathbb{R}$ .
- Check:

$$\frac{d}{dx} \tanh(x) > 0, \quad \forall x,$$

so  $\tanh$  increasing — and injective.

- Also check:

$$\lim_{x \rightarrow \infty} \tanh(x) = 1, \quad \lim_{x \rightarrow -\infty} \tanh(x) = -1.$$

- Thus range =  $(-1, 1)$
- This shows

$$|(-1, 1)| = |\mathbb{R}|,$$

and then use the previous idea that all intervals have same cardinality.

