Proof for Union.

- Let $x > 0$.
  - Then there exists integers larger than x, so the set
    $$S = \{q \in \mathbb{N} : q > x\} \neq \emptyset.$$
  - By Well-ordering, S has a least element. Call it r.
  - Since $r \in S$, $r > x$, and since $r - 1 \notin S$, $r - 1 \geq x$, so we have
    $$r - 1 \leq x < r \implies x \in A_r.$$
- Now let $x < 0$.
  - If x is integer, say $x = -n$, then $x \in A_{-n}$.
  - If x is not an integer, then $-x > 0$ and thus $-x \in A_{r-1}$ as above, or
    $$r - 1 \leq -x < r,$$
    and since x not integer,
    $$r - 1 < -x < r \implies -r < x < -r + 1 \implies x \in A_{-r}.$$