Proof.

Let

$$Q(k) = P(1) \wedge P(2) \wedge \cdots \wedge P(k).$$

Then do regular induction on Q.
$Q(1) = P(1)$ and $P(1)$ is true by assumption.
Assume that $Q(k)$ is true. By the assumptions of the theorem,

$$Q(k) \implies P(k+1),$$

and by the lemma

$$Q(k) \implies Q(k) \wedge P(k+1) \iff Q(k+1).$$

So we have $Q(1)$ is true and $Q(k) \implies Q(k+1)$, so by vanilla induction, $\forall n \in \mathbb{N}, Q(n)$.
But since $Q(n) \implies P(n)$, this gives $\forall n \in \mathbb{N}, P(n)$.

□