Theorem

$$[P(1) \wedge (\forall k \in \mathbb{N}, P(k) \implies P(k+1))] \implies \forall n \in \mathbb{N}, P(n).$$

Proof.

- We will prove by contradiction. Let us assume that the assumption of the theorem is true and the conclusion is false, namely that

  $$P(1) \wedge \Big(\forall k \in \mathbb{N}, P(k) \implies P(k+1)\Big) \wedge \Big(\neg(\forall n \in \mathbb{N}, P(n))\Big).$$

  We also have

  $$\neg(\forall n \in \mathbb{N}, P(n)) \iff \exists n \in \mathbb{N} \neg P(n),$$

  or, if we define $S = \{q \in \mathbb{N} : P(q) \text{ is false } \}$, then $S \neq \emptyset$.
- By the Well-ordering principle, S has a least element ... call it r.
- We have $P(r)$ is false (since $r \in S$) and $P(r-1)$ is true (since $r - 1 \notin S$), but this is a contradiction with $P(r-1) \implies P(r)$.

□