Numerical approximation: Euler's method.

Some times, one may encounter a differential equation that cannot be solved symbolically: e.g.

$$\frac{dy}{dx} = e^{-x^2}$$

solution $y = \int e^{-x^2} dx$ is not an "elementary" function.

How can we calculate such a solution, at least approximately?

Initial value problem

$$\begin{cases} \frac{dy}{dx} = f(x, y(x)) & \leftarrow \text{tells us how } y \text{ changes} \\ y(x_0) = y_0 & \leftarrow \text{tells us where } y \text{ starts.} \end{cases}$$

Euler's method:
- Start at point (x_0, y_0) as specified by initial condition.
- Use Diff. eq. to estimate how y will change over a short interval: At (x_0, y_0), slope is $f(x_0, y_0)$.
  Over the interval x_0 to x_0+h, y will change by approximately $\Delta y = h \cdot f(x_0, y_0)$, so y will be approximately
  $$y_1 = y_0 + h \cdot f(x_0, y_0).$$

[This would be exact if $\frac{dy}{dx}$ were constant on $[x_0, x_0+h]$]

Then repeat starting with $x_1 = x_0+h$ and $y_1 = y_0 + h \cdot f(x_0, y_0)$