We see a pattern: $y_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!}$

so $y(x) = \lim_{n\to\infty} y_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} = e^{-x}$

And indeed, we could have gotten $y = e^{-x}$ as the solution by other methods.

Under what conditions does this actually work?
Depends on $f(x,y)$

Suppose $f(x,y)$ is continuous as a function of two variables
And there is a constant C such that

$$|f(x,y_1) - f(x,y_2)| \le C\,|y_1 - y_2|$$

for all values of x, y_1, y_2

Then the Picard iteration $y = \lim_{n\to\infty} y_n$ always converges to a solution of the IVP $\begin{cases} \frac{dy}{dx} = f(x,y(x)) \\ y(a) = b \end{cases}$

Furthermore $y(x)$ is the unique solution of this IVP.