Sketch: let $y_0(x) = b$

let $y_{n+1}(x) = \underline{P}[y_n](x) = b + \int_a^x f(t, y_n(t))\,dt$

let $y(x) = \lim_{n\to\infty} y_n(x)$ (Have to justify this exists)

Now what is $P[y]$?

$$P[y](x) = b + \int_a^x f(t, y(t))\,dt = b + \int_a^x f(t, \lim_{n\to\infty} y_n(t))\,dt$$

needs justification

$$= \lim_{n\to\infty}\left[b + \int_a^x f(t, y_n(t))\,dt\right] = \lim_{n\to\infty} y_{n+1}(x) = y(x)$$

so $\underline{P}[y] = y$, and y is a genuine solution of $\left\{\begin{array}{l}\frac{dy}{dx} = f(x, y(x)) \\ y(a) = b\end{array}\right\}$!

Example: $\left\{\begin{array}{l}\frac{dy}{dx} = -y \\ y(0) = 1\end{array}\right\}$ $\quad f(x,y) = -y$

$y_0(x) = 1$

$$y_1(x) = P[y_0](x) = 1 + \int_0^x (-y_0(t))\,dt = 1 + \int_0^x (-1)\,dt$$

$$= 1 - x$$

$$y_2(x) = 1 + \int_0^x -y_1(t)\,dt = 1 + \int_0^x -(1-t)\,dt = 1 - x + \frac{x^2}{2}$$

$$y_3(x) = 1 + \int_0^x -\left(1 - t + \frac{t^2}{2}\right)dt = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

$$y_4(x) = 1 + \int_0^x -\left(1 - t + \frac{t^2}{2} - \frac{t^3}{6}\right)dt = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$$