$\underline{\lambda = 0}$: $y = Ax + B$ $\quad y(0) = 0 \Rightarrow B = 0 \quad y'(L) = 0 \Rightarrow A = 0$

$\lambda = 0$ is not an eigenvalue

$\underline{\lambda = \alpha^2 > 0}$: $\quad y = A\cos\alpha x + B\sin\alpha x \qquad y(0) = 0 \Rightarrow A = 0$

so $y = B\sin\alpha x$

$y'(x) = B\alpha\cos\alpha x \qquad y'(L) = 0 \Rightarrow B\alpha\cos\alpha L = 0$

so to be an eigenvalue, must have $\cos\alpha L = 0$.

Thus $\alpha L = \left(n - \frac{1}{2}\right)\pi = \left(\frac{2n-1}{2}\right)\pi \qquad n = 1, 2, 3, \ldots$

$$\alpha_n = \left(\frac{2n-1}{2}\right)\frac{\pi}{L} \qquad \lambda_n = \alpha_n^2 = \left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}\right]^2$$

Eigenfunctions $\quad y_n(x) = \sin\alpha_n x = \sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right]$

Orthogonality: If $n \neq m$, $\displaystyle\int_0^L \sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right]\sin\left[\left(\frac{2m-1}{2}\right)\frac{\pi}{L}x\right]dx = 0$

Eigenfunction series

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right]$$

Where

$$c_n = \frac{\int_0^L f(x)\sin\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right]dx}{\int_0^L \sin^2\left[\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x\right]dx}$$

This is the socalled odd-half multiple sine series for $f(x)$. It comes up, for instance in the heat problem.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = 0 \qquad \frac{\partial u}{\partial t}(0,t) = 0$$

$$u(x,0) = f(x)$$

where the $x=0$ end is held fixed at zero temp, but the $x=L$ end is insulated.