In general, how do we find the coefficients of the eigenfunction series

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) \quad ?$$

Using orthogonality relations to extract a single coefficient.

Consider $\displaystyle\int_a^b f(x) y_m(x) r(x)\, dx = \int_a^b \left(\sum_{n=1}^{\infty} c_n y_n(x) \right) y_m(x) r(x)\, dx$

$$= \int_a^b \sum_{n=1}^{\infty} c_n y_n(x) y_m(x) r(x)\, dx = \sum_{n=1}^{\infty} c_n \int_a^b y_n(x) y_m(x) r(x)\, dx$$

by orthogonality only the $n=m$ term can be nonzero, so

$$= c_m \int_a^b y_m(x) y_m(x) r(x)\, dx .$$

In summary: $\displaystyle\int_a^b f(x)\, y_m(x) r(x)\, dx = c_m \int_a^b [y_m(x)]^2 r(x)\, dx$

Or

$$\boxed{c_m = \frac{\int_a^b f(x) y_m(x) r(x)\, dx}{\int_a^b [y_m(x)]^2 r(x)\, dx}}$$

This generalizes the formulas for b_n and a_n in the sine and cosine series.

<u>Another example</u>: (1) $y'' + \lambda y = 0$ $\qquad p = 1 > 0,\ q = 0 \geq 0,\ r = 1 > 0$

(2) $y(0) = 0$ $\qquad \alpha_1 = 1 \geq 0,\ \alpha_2 = 0 \geq 0$

(3) $y'(L) = 0$ $\qquad \beta_1 = 0 \geq 0,\ \beta_2 = 1 \geq 0$

By main theorem of Sturm Liouville theory, all eigenvalues are nonnegative.