Key examples. (I)

(1) $y'' + \lambda y = 0$ $\quad (p=1, q=0, r=1)$

(2) $y(0) = 0$ $\quad (\alpha_1 = 1, \alpha_2 = 0)$

(3) $y(L) = 0$ $\quad (\beta_1 = 1, \beta_2 = 0)$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad y_n(x) = \sin\frac{n\pi x}{L} \qquad n = 1, 2, 3, \ldots$

Orthogonality: $\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = 0$ if $n \neq m$

Eigenfunction Series: $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ [Sine series]

where $b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$

(II)

(1) $y'' + \lambda y = 0$ $\quad (p=1, q=0, r=1)$

(2) $y'(0) = 0$ $\quad (\alpha_1 = 0, \alpha_2 = 1)$

(3) $y'(L) = 0$ $\quad (\beta_1 = 0, \beta_2 = 1)$

$\lambda_0 = 0, \ \lambda_1 = \left(\frac{\pi}{L}\right)^2, \ \cdots, \ \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 0, 1, 2, \ldots$

$y_0(x) = 1 \qquad y_1(x) = \cos\frac{\pi x}{L}, \ \cdots, \ y_n = \cos\frac{n\pi x}{L}$

Orthogonality: $\int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = 0$ if $n \neq m$

Eigenfunction series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$

where $a_0 = \frac{2}{L}\int_0^L f(x)\, dx, \ a_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\, dx$ [cosine series]