Sturm-Liouville theory 2: Eigenfunction Expasions

Sturm-Liouville problem (1) $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x)y = 0$
on $[a,b]$

(2) $\alpha_1 y(a) - \alpha_2 y'(a) = 0$

(3) $\beta_1 y(b) + \beta_2 y'(b) = 0$

If $p(x) > 0$ and $r(x) > 0$ on $[a,b]$,
We know that the eigenvalues λ form an increasing sequence.

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$$

and that the associated eigenfunctions

$$y_1(x), y_2(x), \ldots, y_n(x), \ldots$$

are unique up to multiplication by a constant.

They also satisfy the orthogonality relation

$$\boxed{\text{If } n \neq m, \text{ then } \int_a^b y_n(x) y_m(x) r(x)\, dx = 0}$$

The last part of the story is that (pretty much) any function $f(x)$ defined on $[a,b]$ can be written as a series in the Eigenfunctions:

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

This is called the <u>eigenfunction series</u> for $f(x)$ coming from the Sturm-Liouville problem.

Thus for each function $f(x)$, there are <u>many</u> ways to write it as a series, one for each Sturm-Liouville problem.