Sturm-Liouville problems and eigenfunction series

In heat, wave, and Laplace's equation, separation of variables often leads to problems like

$$\frac{d^2y}{dx^2} + \lambda y = 0 \quad , \quad y(0) = 0 \quad , \quad y(L) = 0$$

The eigenfunctions are $\sin \frac{n\pi x}{L}$.

Usually the initial data has to be expressed as a sine series. Can we generalize this?

Sturm-Liouville equations:

A second order linear equation with parameter λ is in Sturm-Liouville standard form if it looks like

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x)y = 0 \qquad \text{(SL form)}$$

Fact: any second order linear equation

$$A(x)y'' + B(x)y' + C(x)y + \lambda D(x)y = 0$$

can be put in SL standard form

Main example: $\frac{dy^2}{dx^2} + \lambda y = 0$ $\quad p(x) = 1 \quad q(x) = 0 \quad r(x) = 1$

is in SL standard form.

Other example: $\frac{d}{dx}\left[x\frac{dy}{dx}\right] - \frac{n^2}{x}y + \lambda x y = 0$

Bessel equation (Laplace eqn in cylindrical coords)