$$u(0,y)=0 \Rightarrow X(0)Y(y)=0 \Rightarrow X(0)=0$$
$$u(a,y)=0 \Rightarrow X(a)Y(y)=0 \Rightarrow X(a)=0$$
$$u(x,0)=0 \Rightarrow X(x)Y(0)=0 \Rightarrow Y(0)=0$$

So we get
$$\underbrace{\frac{d^2X}{dx^2}+\lambda X=0 \qquad X(0)=0 \qquad X(a)=0}_{\text{That same eigenvalue problem!}} \qquad\qquad \frac{d^2Y}{dy^2}-\lambda Y=0 \qquad Y(0)=0$$

Eigenvalues: $\lambda_n=\left(\frac{n\pi}{a}\right)^2 \quad n=1,2,3,\dots$

Eigenfunctions: $X_n(x)=\sin\frac{n\pi x}{a}$

What is the Y_n- partner for X_n?

$\lambda_n=\left(\frac{n\pi}{a}\right)^2$ so Y_n satisfies $\frac{d^2Y_n}{dy^2}-\left(\frac{n\pi}{a}\right)^2Y_n=0$

characteristic eqn $r^2-\left(\frac{n\pi}{a}\right)^2=0 \Rightarrow r=\pm\frac{n\pi}{a}$

so $Y_n(y)=c_1e^{\frac{n\pi y}{a}}+c_2e^{-\frac{n\pi y}{a}}$

We can now use the condition $Y(0)=0$:

$$0=Y_n(0)=c_1e^0+c_2e^0=c_1+c_2$$

Thus $c_2=-c_1$, and we can write

$$Y_n=c_1\left(e^{\frac{n\pi y}{a}}-e^{-\frac{n\pi y}{a}}\right)=2c_1\left(\frac{e^{\frac{n\pi y}{a}}-e^{-\frac{n\pi y}{a}}}{2}\right)$$
$$=2c_1\sinh\left(\frac{n\pi y}{a}\right) \quad \text{hyperbolic sine.}$$