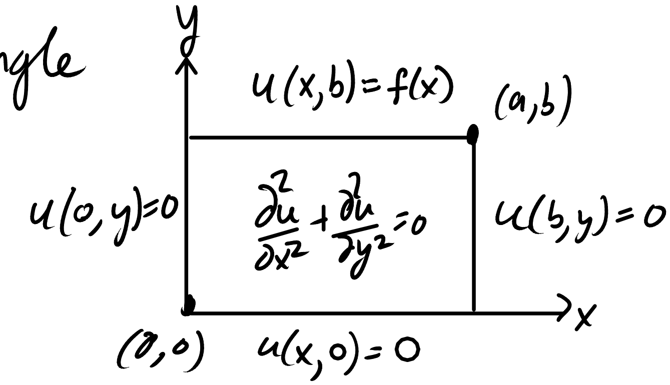
Laplace's Equation on a rectangle

Today we will solve the problem

Homogeneous boundary conditions on all sides except one.



Separation of variables: Separable solutions $u(x,y) = X(x)\,Y(y)$

Laplace eqn: $$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{d^2 X}{dx^2}\,Y + X\,\frac{d^2 Y}{dy^2}$$

Divide by $X\,Y$: $$0 = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2}$$

So $$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{only depends on } x} = \underbrace{-\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{only depends on } y} \implies \text{both sides constant} = -\lambda$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda = -\frac{1}{Y}\frac{d^2Y}{dy^2} \implies \begin{aligned} &\frac{d^2X}{dx^2} + \lambda X = 0 \\ &\frac{d^2Y}{dy^2} - \lambda Y = 0 \end{aligned}$$

Boundary conditions:

$$u(x,0) = 0$$
$$u(x,b) = f(x)$$
$$u(0,y) = 0$$
$$u(a,y) = 0$$

Let's forget about this one for awhile. We first work with the homogeneous conditions.