6) Initial conditions

Initial position: $u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \stackrel{?}{=} f(x)$

This is the Fourier sine series for $f(x)$.

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Velocity: $\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left[A_n \frac{cn\pi}{L}\left(-\sin \frac{cn\pi t}{L}\right) + B_n \frac{cn\pi}{L} \cos \frac{cn\pi t}{L} \right] \sin \frac{n\pi x}{L}$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \left(B_n \frac{cn\pi}{L}\right) \sin \frac{n\pi x}{L} = g(x)$$

This is a Fourier sine series for $g(x)$, but with a twist.

$B_n \frac{cn\pi}{L}$ = Fourier sine coefficient of $g(x)$ $= \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$

$$B_n = \left(\frac{L}{cn\pi}\right) \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$$

Special cases · Plucked string (like guitar) $u(x,0) = f(x)$

$\frac{\partial u}{\partial t}(x,0) = 0$

$u(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{cn\pi t}{L} \sin \frac{n\pi x}{L}$, where $A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$

· Struck string (like piano) $u(x,0) = 0$

$\frac{\partial u}{\partial t}(x,0) = g(x)$

$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{cn\pi t}{L} \sin \frac{n\pi x}{L}$ where $B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$