2) Boundary conditions $u(0,t)=0 \Rightarrow X(0)T(t)=0 \Rightarrow X(0)=0$

$u(L,t)=0 \Rightarrow X(L)T(t)=0 \Rightarrow X(L)=0$

So $X(x)$ satisfies

$$\left.\begin{array}{l} \dfrac{d^2X}{dx^2}+\lambda X=0 \\ X(0)=0 \\ X(L)=0 \end{array}\right\}$$

A "well-known" eigenvalue problem.

3) Solution: eigenvalues $\left(\frac{\pi}{L}\right)^2, \left(\frac{2\pi}{L}\right)^2, \ldots \quad \lambda_n=\left(\frac{n\pi}{L}\right)^2 \quad n=1,2,3,\ldots$

eigenfunctions (Modes of oscillation. $\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots \quad X_n=\sin\frac{n\pi x}{L} \quad n=1,2,3.$

4) T_n must satisfy $\dfrac{d^2T_n}{dt^2}+c^2\lambda_n T_n=0$

$$\frac{d^2T_n}{dt^2}+c^2\left(\frac{n\pi}{L}\right)^2 T_n=0$$

Solutions: $\sin\frac{cn\pi t}{L}, \cos\frac{cn\pi t}{L}$

$$T_n(t)=A_n\cos\frac{cn\pi t}{L}+B_n\sin\frac{cn\pi t}{L}$$

So $u_n(x,t)=\left(A_n\cos\frac{cn\pi t}{L}+B_n\sin\frac{cn\pi t}{L}\right)\sin\frac{n\pi x}{L}$

5) The conditions $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$, $u(0,t)=0$, $u(L,t)=0$

are linear homogeneous, so the general solution is

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{cn\pi t}{L}+B_n\sin\frac{cn\pi t}{L}\right)\sin\frac{n\pi x}{L}$$