Each eigenfunction yields a solution to the heat equation

$$X_n(x) = \sin \frac{n\pi x}{L} \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

What is the T counterpart for X_n?

It should satisfy

$$\frac{dT}{dt} = -k\lambda_n T = -k\left(\frac{n\pi}{L}\right)^2 T$$

Therefore

$$T(t) = Ce^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad C \text{ constant}$$

Put it together:

$$u(x,t) = T(t)X_n(x) = Ce^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi x}{L}$$

Satisfies
$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} & \text{heat eqn} \\ u(0,t) = 0 & \text{Boundary} \\ u(L,t) = 0 & \text{conditions} \end{cases}$$

Another type of boundary condition $\quad \frac{\partial u}{\partial x}(0,t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = 0$

"insulated ends"

$$\left(\frac{\partial u}{\partial x} = \text{heat flux} \right) \qquad \frac{dX}{dx}(0)T(t) = 0 \qquad \frac{dX}{dx}(L)T(t) = 0$$

So $X(x)$ satisfies
$$\begin{Bmatrix} X'' + \lambda X = 0 \\ X'(0) = 0 \\ X'(L) = 0 \end{Bmatrix}$$

Eigenvalues: $\lambda_0 = 0$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $\quad n = 1,2,3,\ldots$

Eigenfunctions: $X_0 = 1$, $X_n = \cos \frac{n\pi x}{L}$ $\quad n = 1,2,3,\ldots$

Heat solutions: $u_0(x,t) = 1$, $u_n(x,t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos \frac{n\pi x}{L}$