In terms of the factorizable function $u(x,t) = X(x)T(t)$

This means $u(0,t) = X(0)T(t) = 0$

$u(L,t) = X(L)T(t) = 0$

So we need $\left.\begin{array}{l} X(0)=0 \\ X(L)=0 \end{array}\right\}$ Endpoint conditions for $X(x)$

Thus X satisfies the endpoint value problem

$$\left\{\begin{array}{ll} \dfrac{d^2X}{dx^2} + \lambda X = 0 & (1) \\ X(0) = 0 & (2) \\ X(L) = 0 & (3) \end{array}\right.$$

This is the eigenvalue problem we studied earlier!

Recall how to find positive eigenvalues and eigenfunctions

$$\frac{d^2X}{dx^2} + \lambda X = 0 \implies X(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$$

$$X(0) = 0 \implies A = 0 \implies X(x) = B\sin\sqrt{\lambda}\,x$$

$$X(L) = 0 \implies B\sin\sqrt{\lambda}\,L = 0$$

X can only be "interesting"/nontrivial if $B \neq 0$, so we must have $\sin\sqrt{\lambda}\,L = 0 \implies \sqrt{\lambda}\,L = n\pi \quad n = 1,2,3,\ldots$

$$\implies \lambda = \left(\frac{n\pi}{L}\right)^2 \quad n = 1,2,3,\ldots$$

It turns out $\lambda < 0$ and $\lambda = 0$ are <u>not</u> eigenvalues.

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1,2,3,\ldots$

Eigenfunctions: $X_n(x) = \sin\frac{n\pi x}{L}$