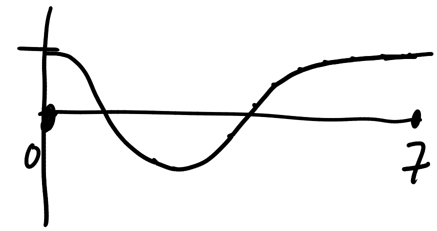
We can also study several variations on this problem
We'll keep the differential equation the same, but change the endpoint conditions:

Vanishing derivative at endpoints

$$\begin{cases} y'' + \lambda y = 0 & (1) \\ y'(0) = 0 & (2) \\ y'(7) = 0 & (3) \end{cases}$$




curve should have horizontal tangent at endpoints.

What are eigenvalues and eigenfunctions?
Consider $\lambda > 0$.

$$(1)\ y'' + \lambda y = 0 \Rightarrow y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$$

$$y' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x$$

$$(2)\ y'(0) = 0 \Rightarrow -c_1\sqrt{\lambda}\cdot 0 + c_2\sqrt{\lambda}\cdot 1 = 0$$

$$\Rightarrow c_2 = 0$$

So $y = c_1 \cos\sqrt{\lambda}\, x$

What about (3)? $y'(7) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\cdot 7) = 0$
This will force $c_1 = 0$ unless $\sin(\sqrt{\lambda}\cdot 7) = 0$

$\sin(\sqrt{\lambda}\cdot 7) = 0 \iff \sqrt{\lambda}\cdot 7 = n\pi$ for some integer $n$

$$\iff \lambda = \left(\frac{n\pi}{7}\right)^2$$