Case $\lambda > 0$: $\quad y'' + \lambda y = 0 \Rightarrow y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$

$y(0) = 0 \Rightarrow c_1 \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0$

so $y = c_2 \sin\sqrt{\lambda}\,x$

$y(L) = 0 \Rightarrow c_2 \sin\sqrt{\lambda}\,L = 0$

Now either $c_2 = 0$ or $\sin\sqrt{\lambda}\,L = 0$

$\sin\sqrt{\lambda}\,L = 0$ happens if $\sqrt{\lambda}\,L = n\pi$ for some integer n.

$$\sqrt{\lambda} = \frac{n\pi}{L} \qquad \frac{\sqrt{\lambda}\,L}{\pi} = n$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

So there are two possibilities:

- if $\frac{\sqrt{\lambda}\,L}{\pi}$ is an integer then there are infinitely many sol'ns

$$y = c_2 \sin\sqrt{\lambda}\,x = c_2 \sin\frac{n\pi}{L}x$$

- if $\frac{\sqrt{\lambda}\,L}{\pi}$ is not an integer, then there is only one solution

$$y = 0$$

If $\lambda < 0$, we can show that $\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(L) = 0 \end{cases}$ has only $y = 0$ as a solution.

The numbers $\lambda = \left(\frac{n\pi}{L}\right)^2$ for which there are infinitely many solutions are called the eigenvalues, and the nonzero solutions: $y = \sin\frac{n\pi}{L}x$ are the eigenfunctions.