We expect a "steady periodic" solution, that also has a Fourier series

$$x_{sp}(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi t}{L} + B_n \sin \frac{n\pi t}{L} \right)$$

Now the Fourier coefficients of $x_{sp}(t)$ play the role of "undetermined coefficients".

Plug this undetermined series into $mx'' + kx$

$$\frac{kA_0}{2} + \sum_{n=1}^{\infty} \left[k - m\left(\frac{n\pi}{L}\right)^2\right] A_n \cos \frac{n\pi t}{L} + \left[k - m\left(\frac{n\pi}{L}\right)^2\right] B_n \sin \frac{n\pi t}{L}$$

Want this equals $F(t)$, so we need

$$\frac{kA_0}{2} = \frac{a_0}{2} \qquad A_0 = \frac{a_0}{k}$$

$$\left[k - m\left(\frac{n\pi}{L}\right)^2\right] A_n = a_n \qquad A_n = \frac{a_n}{\left[k - m\left(\frac{n\pi}{L}\right)^2\right]}$$

$$\left[k - m\left(\frac{n\pi}{L}\right)^2\right] B_n = b_n \qquad B_n = \frac{b_n}{\left[k - m\left(\frac{n\pi}{L}\right)^2\right]}$$

This works as long as none of the denominators

$k - m\left(\frac{n\pi}{L}\right)^2$ equals zero. Otherwise, resonance occurs.